Vibration Analysis of Multi-Symmetric Structures

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This paper demonstrates that for multi-symmetric structures, symmetry concepts can be used to reduce the size of the eigenvalue problem which must be modeled and solved using a discrete formulation. It also demonstrates that an understanding of group theory can be used to order and classify the vibration modes of multi-symmetric structures. These ideas have been available to workers in quantum mechanics and molecular vibrations since the 1930's but they have not been appreciated and understood by structural analysts. The primary new innovation of this report is the demonstration that it is not necessary to formulate a structural dynamic model for more than a small pie-shaped segment of the structure. This innovation required the development of appropriate boundary conditions on the edges of the pie-shaped segment, and these have been derived for an equilateral triangle and a hexagon. Although these ideas are demonstrated herein for flat polygonal membranes, they can be generalized to space frames, dish antennas, and large three-dimensional structures. From a practical standpoint, the application of these ideas can significantly reduce the dimension (i.e., size) of the eigenvalue problem which must be solved to compute the vibration modes of multi-symmetric structures.

Introduction

THIS study, termed "Multi-Symmetric Structural Analysis," stems from a long-standing interest in symmetry concepts and the uses of symmetry. In 1964, structural dynamicists at NASA Langley were involved in computing the modes and frequencies of the Saturn-I launch vehicle (Fig. 1). The first-stage booster was basically a central cylinder surrounded by eight symmetrically placed outer propellant tanks in an octagonal arrangement. Several contractors analyzed the vibration modes of the Saturn-I vehicle, with each analyst choosing a different coordinate system and making a different use of the symmetry inherent in the structure.

This technical problem generated an interest in symmetry and symmetry concepts at NASA Langley, and it led eventually to (unpublished) studies by two of the author's colleagues, J.S. Mixon and J.T. Howlett of NASA Langley Research Center. Mixon's work[1] was prompted by Mary Waller's excellent book on plate vibrations.[2] Howlett's work[3] was prompted by an interest in group theory and its applications.[4]

At TRW, an interest in symmetry and its applications arose when it was desired to compute the acoustic modes within a star-shaped solid-propellant rocket cavity (Fig. 2). Other practical applications of symmetry concepts include such things as an n-bladed helicopter rotor where it is desired to calculate the vibration modes of the rotor. Another practical application is an n-ribbed spacecraft antenna where it is desired to compute vibration modes or static response.

The ideas and symmetry concepts presented herein pertain in many respects to *all* these practical problems just discussed. From a practical standpoint, the application of symmetry concepts allows the analyst to compute vibrational modes and frequencies (or buckling modes and engenvalues) using fewer *degrees-of-freedom* than is commonly employed. This fact allows the analyst to model only 1/6, 1/8, (or 1/nth) of the structure, as opposed to using 1/2, or 1/4th as is generally the case presently.

The net result is a savings in two areas: 1) the size of the discrete (e.g., finite-element) model which must be constructed can be made smaller, and 2) the "size" (e.g., degrees-of-freedom) of the eigenvalue problem (or response problem) is reduced. Sometimes these savings can represent a factor of 2, 3, or more, depending upon the problem involved.

Simple Examples of Symmetry

Consider a singly symmetric structure, such as the tapered beam, delta wing airplane, etc; shown in Fig. 3. These figures all possess a *single plane of symmetry*. Consequently, the dynamic matrix

$$[K - M\omega^2]w \tag{1}$$

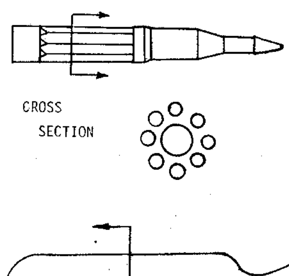

Fig. 1 Saturn-I launch vehicle with symmetrically located tanks.

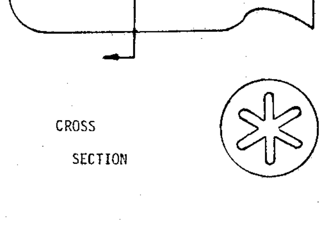

Fig. 2 Acoustic cavity of a solid propellant rocket motor.

Presented as Paper 75-808 at the AIAA/ASME/SAE 16th Structures, Structural Dynamics and Materials Conference, Denver, Colo., May 27-29, 1975; submitted May 30, 1975; revision received September 26, 1975. This paper represents the results of work performed largely in the author's spare time during the past ten years. As such, many discussions have been held with colleagues at NASA Langley and TRW Systems during this time, and many individuals too numerous to mention have shed light on the vibrations of multi-symmetric structures. However, I wish to thank J.S. Mixon of NASA/Langley, P.G. Bhuta of TRW Systems, and J. Matthews of the Physics Department of California Institute of Technology for their help, guidance, support, and suggestions concerning the work reported herein.

Index category: Structural Dynamic Analysis.

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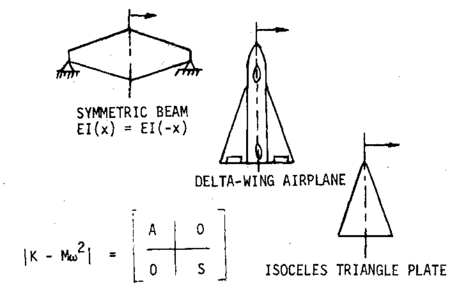

Fig. 3 Structures with a single plane of symmetry.

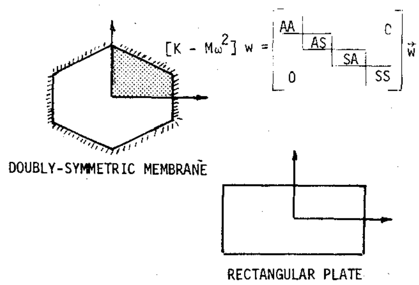

Fig. 4 Structures with two planes of symmetry.

can be transformed to "block diagonal form"

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & S \end{array}\right] \{w\} = 0 \tag{2}$$

where the matrix $[A]$ provides the antisymmetric modes, and the matrix $[S]$, the symmetric modes.

Armed with this knowledge, the structural analyst just models one-half of the structure (e.g., the shaded portion, Fig. 3). Then, to this discrete model, the analyst appends the "boundary condition"

$$w = 0 \text{ (antisymmetric)} \tag{3}$$

on the plane of symmetry and thereby generates the eigenvalue problem

$$[A]w = 0 \tag{4}$$

for the antisymmetric modes.

Similarly, he must also apply the "boundary condition"

$$\nabla w \cdot n = \partial w / \partial n = 0 \text{ (symmetric)} \tag{5}$$

$$[S]w = 0 \tag{6}$$

for the symmetric modes.

Now consider a doubly symmetric structure such as the rectangular plate and doubly symmetric membrane shown in Fig. 4. The rectangular plate and the membrane each possess *two planes of symmetry,* and the dynamic matrix can be transformed to "block diagonal form" again

$$[K - M\omega^2]w = \left[\begin{array}{c|c|c|c} AA & & & 0 \\ \hline & AS & & \\ \hline & & SA & \\ \hline 0 & & & SS \end{array}\right] w = 0 \tag{7}$$

by the proper choice of unknowns.

In actual practice, the structural analyst models just one-quadrant (e.g., the shaded area in Fig. 4) of the structure and applies the "boundary conditions" of symmetry or antisymmetry on the x and y axes. The practice just described of modeling and solving one-quadrant of a doubly symmetric structure is a well-established technique used by structural analysts.

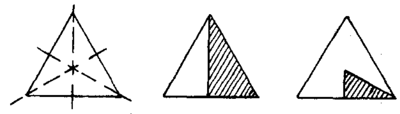

Fig. 5 Equilateral triangular membrane: a) three planes of symmetry; b) conventional approach to modelling; and c) multi-symmetric approach.

Now consider the equilateral triangular membrane shown in Fig. 5. Now there are three planes of symmetry, which intersect in an axis through the centroid of the triangle. The question is "How much of the structure it is necessary to model?" One-half? One-third? Perhaps just one-sixth? One approach to answering these questions is given in the section which follows.

Vibrations of an Equilateral Triangular Membrane

In studying transverse vibrations of an equilateral triangular membrane using a discrete formulation (e.g., a finite-element technique) the classical eigenvalue problem

$$[K - M\omega^2]w = 0 \tag{8}$$

is encountered.

As described previously[5] the conventional approach to this problem is to model one-half the structure and apply conditions of symmetry or antisymmetry across the median plane which bisects the triangular platform.

With this approach, Eq. (8) becomes partitioned as

$$[K - M^2]w = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & S \end{array}\right] w = 0 \tag{9}$$

where the antisymmetric modes are computed using the matrix $[A]$, and the symmetric modes are computed using the matrix $[A]$, and the symmetric modes are computed using $[S]$. As reported in Ref. 5, the small finite-element example therein involved

$$[A] = [12 \times 12]$$

$$[S] = [19 \times 19]$$

and the full matrix is

$$\left[\begin{array}{c|c} 12 \times 12 & 0 \\ \hline 0 & 19 \times 19 \end{array}\right] = [K - M\omega^2] \tag{10}$$

In 1930, Wigner[6] used group theory to show that the physical symmetry inherent in some structures, such as an equilateral triangle, could be used to 1) classify the modes of vibration, and 2) predict degeneracies which would occur (i.e., degenerate modes).

Wigner's classic paper is now available in English in Cracknell's text.[7] Using Wigner's approach, the eigenvalue problem $[K - M\omega^2]$ can be transformed to "block diagonal form," as discussed by Rosenthal and Murphy in 1936.[8] For the problem at hand, we have

$$[\theta]^T [K - M\omega^2] \; [\theta] w = [K' - M'\omega^2] w'$$

$$= \left[\begin{array}{c|c|c|c} AA & & & 0 \\ \hline & SS & & \\ \hline & & D & \\ \hline 0 & & & D \end{array}\right] w' = 0 \tag{11}$$

where $[\theta]$ is an orthogonal transformation

$$[\theta]^T[\theta] = [I] = [\theta]^{-1}[\theta] \tag{12}$$

which can be found using group theory, and the individual matrices are $[AA]$ for the (antisymmetric) (antisymmetric) modes, $[SS]$ for the (symmetric) (symmetric) modes, and $[D]$ for the degenerate modes.

Using the example of Ref. 5, we find

$$[AA] = [2\times 2]$$

$$[SS] = [9\times 9]$$

$$[D] = [10\times 10]$$

and thus, Eq. (11) gives

$$\left[\begin{array}{c|c|c|c} 2\times 2 & & 0 & \\ \hline & 9\times 9 & & \\ \hline & & 10\times 10 & \\ \hline 0 & & & 10\times 10 \end{array}\right] w' = 0 \tag{13}$$

for the same 31 "modes" as contained in Eq. (10), i.e, the conventional approach. However, in the conventional approach the degenerate modes are buried (i.e., hidden) in the two matrices $[A]$ and $[S]$ of Eq. (9).

Although this approach using the orthogonal transformation has been implemented[3] an easier method is as follows:

First build a discrete moddel of one-sixth of the structure,[5] and then by applying appropriate boundary conditions generate *three* eigenvalue problems, namely,

$$[AA]w = 0 \tag{14a}$$

$$[SS]w = 0 \tag{14b}$$

$$[D]w = 0 \tag{14c}$$

corresponding to Eq. (11). In our current example, Eqs. (14) become

$$[2\times 2]w = 0 \tag{15a}$$

for the "totally symmetric," i.e. (symmetric) (symmetric) modes, and

$$[9\times 9]w = 0 \tag{15b}$$

for the "totally symmetric," i.e. (symmetric) (symmetric) modes, and

$$[10\times 10]w = 0 \tag{15c}$$

for the 10 "degenerate" modes. As discussed in Ref. 5, this approach requires that a finite-element model (i.e., a discrete model) be constructed for only 1/6 the structure in the present example.

The key elements in this method are clearly the boundary condition(s) to be applied along the edge of the small 30-60-90 triangle, Fig. 5. These boundary conditions are straightforward in the case of the $[AA]$ and $[S]$ modes, but they are more involved for the degenerate $[D]$ modes.

The modes themselves can be visualized most readily for the continous system (as opposed to the discrete model) and these modes are sketched roughly in Figs. 6-8. Referring to Fig. 6, the nodal lines of the vibrating membrane coincide with the three medians of the equilateral triangle. The small individual 30-60-90 triangles each oscillate with opposite phase and thus alternate in sign from plus (+) to minux (−) when one moves

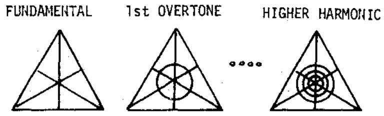

Fig. 6 Nodal patterns for the $[AA]$ modes.

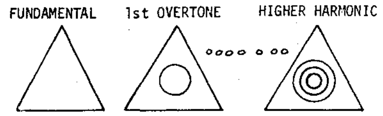

Fig. 7 Nodal patterns for the $[SS]$ modes.

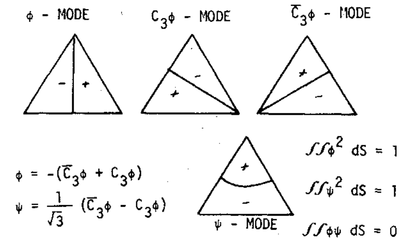

Fig. 8 Degenerate modes ϕ, $C_3\phi$, $\bar{C}_3\phi$, and ψ.

"circumferentially" around the triangle. Figure 6a shows the fundamental mode of this $[AA]$ symmetry class, and Fig. 6b shows the first overtone (of this class). By sketching in more "nodal circles," we can readily envision the higher harmonics of this class of modes.

It is clear from this discussion that the description $[AA]$ (i.e., "anti-anti") was chosen to describe the modes which were *antisymmetric* on *both* boundaries of the 30-60-90 triangle described previously. Similarly, it is apparent that the boundary condition

$$\partial w/\partial n = \nabla w\cdot n = 0 \tag{16}$$

of symmetry (as opposed to antisymmetry) can be applied to both boundaries of the discrete model and thereby generate modes which are "symmetric-symmetric", Eq. (14b).

To generate the matrix $[SS]$ for this case, we must use the symmetry condition, Eq. (16). The corresponding mode shapes (for the continuous membrane) are sketched in Fig. 7. This class of modes $[SS]$ includes the fundamental mode of the membrane, i.e., the lowest frequency mode.

Finally, we come to the heart of the problem, namely the degenerate modes. By applying appropriate boundary conditions for one final time, we wish to generate the matrix $[D]$ of Eqs. (14c) and (15c). These boundary conditions involve some understanding of symmetry and in particular the point space group C_{3v} (see Cracknell).[7] The ideas are fairly straightforward, and their derivation is again facilitated by referring to the continuous system (as opposed to the discrete model).

Referring to Fig. 8, we know from long experience (with airplanes, delta wings, etc.) that the antisymmetric mode "ϕ" of Fig. 8a represents one mode of the membrane. Similarly, if we "rotate the mode" we obtain Figs. 8b and 8c, and denote these modes by $C_3\phi$ and $\bar{C}_3\phi$, respectively. The notation C_3 and $\bar{C}_3$ are borrowed from group theory and denote rotations of $+120°$ (clockwise) and $-120°$ (counterclockwise), respectively.

Now it is clear that the "companion modes" ϕ, $C_3\phi$, and $\bar{C}_3\phi$ cannot *all* be independent. In fact, they are degenerate, and the degeneracy is of order 2†. That is, any *two* of the three "companions" are independent, and we find it convenient to choose the two orthogonal combinations

$$\phi = -(C_3\phi + \bar{C}_3\phi) \text{ (antisymmetric)} \tag{17a}$$

and

$$\psi = \frac{1}{\sqrt{3}}(\bar{C}_3\phi - C_3\phi) \text{ (symmetric)} \tag{17b}$$

†cf. Eq. (11) in which the matrix $[D]$ appears twice.

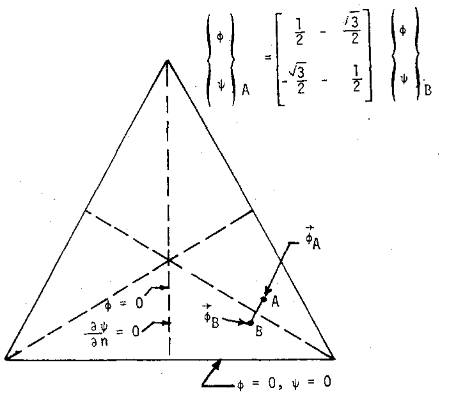

Fig. 9 Boundary conditions on the 30-60-90 triangle.

These combinations have been chosen such that when integrated over the area of the full equilateral triangle, they satisfy the orthogonality relations

$$\iint \phi^2 dS = 1 \tag{18a}$$

$$\iint \psi^2 dS = 1 \tag{18b}$$

$$\iint \phi\psi dS = 0 \tag{18c}$$

The modes $\phi$and ψ thus specified are degenerate, and consequently *both* have exatly the same eigenvalue λ. In fact, it is clear that once we calculate the mode ϕ, all we need to do is to rotate it and superimpose it properly to obtain its orthogonal companion, mode ψ.

Now, to return to the problem at hand, we wish to specify appropriate boundary conditions on the small 30-60-90 triangle such that we can calculate ϕ (and ψ) with a minimim-sized finite-element model.

Referring to Fig. 9, the boundary conditions are

$$\phi = 0$$

and

$$\nabla\psi\cdot\boldsymbol{n} = \partial\psi/\partial n = 0 \tag{19}$$

along the vertical axis (the y-axis) and along the diagonal boundary we have

$$\phi + \sqrt{3}\psi = 0 \tag{20a}$$

and

$$\nabla(\psi - \sqrt{3}\phi)\cdot\boldsymbol{n} = 0 \tag{20b}$$

Now define a vector by

$$\boldsymbol{\phi} = \begin{Bmatrix}\phi\\ \psi\end{Bmatrix} \tag{21}$$

It is clear that if we calculate the modes ϕ and ψ within the 30-60-90 triangle (namely at point B in Fig. 9), we must then find some way to transfer the solution outside the boundary (namely to point A). Writing Eq. (20) as

$$[M_a]\boldsymbol{\phi}_A = [M_b]\boldsymbol{\phi}_B \tag{22a}$$

(where $[M_a]$ and $[M_b]$ are the matrices involved) we can then invert $[M_a]$ to obtain

$$\boldsymbol{\phi}_A = [M_a]^{-1}[M_b]\boldsymbol{\phi}_B \tag{22b}$$

which thus provides the desired relation

$$\begin{Bmatrix}\phi\\ \psi\end{Bmatrix}_A = \begin{bmatrix}1/2 - (3^{1/2}/2)\\ -(3^{1/2}/2) - 1/2\end{bmatrix}\begin{Bmatrix}\phi\\ \psi\end{Bmatrix}_B \tag{23}$$

for continuing the solution across the boundary from point B to point A.

The reader familiar with group theory will recognize the matrices $[M_a]$, $[M_b]$, etc., as two-dimensional representations of the elements C_3 and $\bar{C}_3]$, etc., as two-dimensional representations of the elements C_3 and $\bar{C}_3$ of the group C_{3v}. Similarly, he will have little difficulty in deriving Eqs. (20) for the boundary conditions, as shown in Appendix A.

Vibrations of a Hexagonal Membrane

Several years ago, a colleague of mine became interested in the vibrations of a hexagonal plate. His study was prompted by reading Mary Waller's book on plate vibrations and Chaldni sand patterns.[2] Mixson[1] performed experiments where he recorded the nodal patterns of a free hexagonal plate, and he subsequently calculated the vibration modes using a convenient finite-difference computer program.[9]

In the course of his work, Mixson prepared a table of nodal patterns (i.e., modes) and frequencies for the plate. We were both interested in symmetry, degenerate modes, repeated eigenvalues, etc., at that time, and we searched for some way to classify and catalog his results, but to no avail. However, recent studies on the vibration of symmetrical membranes resulted in another look at this old problem and led to the present work.

For those who think this problem too academic and divorced from applications, it should be noted that many of the results are analogous to the vibrations of a hexagonal spacecraft antenna. In fact, if the ultimate goal is to apply the concepts of multiple symmetry to calculate vibrations of real-world structures, a strong case can be made for first applying the procedures to a problem that is readily grasped, like the transverse vibrations of a hexagonal membrane.

By referring to Cracknell's text (Ref. 7, p. 269) we find that for hexagonal symmetry there are "four one-dimensional and two two-dimensional representations of the hexagonal rotation group." For our purposes, this means that (with a proper choice of generalized coordinates) the dynamic matrix

$$[K - M\omega^2]w = 0 \tag{24}$$

can be factored into block diagonal form as shown following:

$$\begin{bmatrix} \Gamma_1 & & & & & & & 0 \\ & \Gamma_2 & & & & & & \\ & & \Gamma_3 & & & & & \\ & & & \Gamma_4 & & & & \\ & & & & \Gamma_5 & & & \\ & & & & & \Gamma_5 & & \\ & & & & & & \Gamma_6 & \\ 0 & & & & & & & \Gamma_6 \end{bmatrix} w' = 0 \tag{25}$$

The matrices Γ_1, Γ_2, Γ_3, Γ_4 each appear just once in Eq. (25) and are associated with the irreducible representations‡ of the four one-dimensional representations of the hexagonal rotation group. The matrix Γ_5 appears twice (i.e., in two blocks) and the matrix Γ_6 also appears twice in Eq. (25); these matrices are associated with the irreducible representations of the two two-dimensional representations of the hexagonal rotation group. For a discussion of the steps involved in transforming the dynamic matrix to block diagonal form, the interested reader may wish to consult the text by Wilson, Decius, and Cross.[11]

In the terminology of structural dynamics, the fact that Eq. (24) can be transformed to the "block diagonal form" of Eq. (25) means that the modes are "uncoupled." From a practical standpoint, this means that each class of modes can be computed from a smaller eigenvalue problem.

If we rewrite Eq. (25) in a form more akin to the jargon of the structural dynamicist, we have

$$\begin{bmatrix} AA & & & & & & & \\ & SS & & & & & 0 & \\ & & AS & & & & & \\ & & & SA & & & & \\ & & & & D_1 & & & \\ & & & & & D_1 & & \\ & 0 & & & & & D_2 & \\ & & & & & & & D_2 \end{bmatrix} w = 0 \qquad (26)$$

The modes which result by applying conditions of symmetry and antisymmetry along the boundaries of the wedge-shaped sector are shown in Figs. 10 through 13. The first pair of degenerate modes satisfies

$$\phi = (\bar{C}_6\phi + C_6\phi) \text{ (antisymmetric in } y) \qquad (27a)$$

$$\psi = 1/\sqrt{3}\,(C_6\phi - \bar{C}_6\phi) \text{ (symmetric in } y) \qquad (27b)$$

in direct analogy with the equilateral triangle discussed previously.

Similarly, the second pair of degenerate modes satisfies

$$\phi' = -(C_6\phi' + \bar{C}_6\phi) \qquad (28a)$$

$$\psi' = 1/\sqrt{3}\,(\bar{C}_6\phi' - C_6\phi') \qquad (28b)$$

These degenerate pairs and their overtones are sketched in Figs. 14 and 15. Boundary conditions necessary to generate these degenerate modes from a wedge-shaped sector are similar to Eqs. (20) and can be found in Ref. 5. As discussed in Ref. 5, the classes of modes illustrated herein for the hexagon allowed a symmetric classification of the modes Nixson calculated for a hexagonal plate.

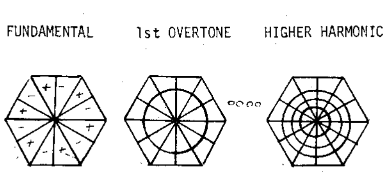

Fig. 10 Modes of the [AA] symmetry class.

‡The term irreducible representation is from group theory and is discussed in Refs. 4 and 6.

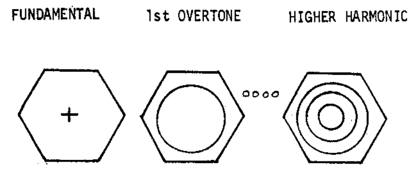

Fig. 11 Modes of the [SS] symmetry class.

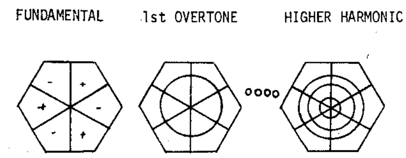

Fig. 12 Modes of the [AS] symmetry class.

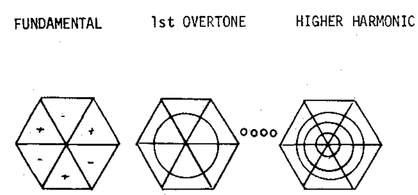

Fig. 13 Modes of the [SA] symmetry class.

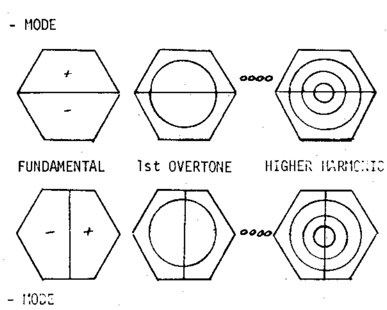

Fig. 14 Sketch of degenerate ϕ-ψ modes and their harmonics.

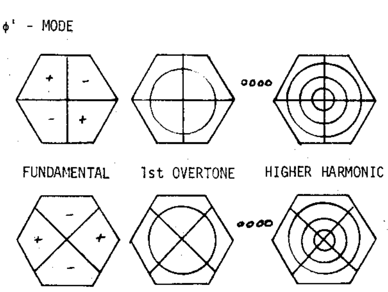

Fig. 15 Sketch of degenerate ϕ'-ψ' modes and their harmonics.

Generalizations of the Analysis

One criticism of the preceding discussion is that the analysis has been limited thus far to transferse vibrations of planar structures such as plates and membranes. However, the analysis can be generalized to include the vibrations (or buckling) or curves structures (e.g., dish antennas), 3-dimensional space frames (e.g., such as the large 90-ft dish at the Goldstone Tracking station), or solid elastic bodies.

Finally, if the total structure is not multisymmetric, but a major substructure (such as the dish of the antenna) is indeed multisymmetric, then the well-known technique of component modes developed by Hurty[12] can be used, wherein multisymmetric analysis is employed to select and find the modes of the substructure. Indeed, large dish antennas are of themselves quite nearly symmetric, but they are often supported at only three or four gimbal points which degrades their symmetry. The combined techniques of component modes and multiple-symmetry may prove useful for future analysis of such structures.

Although the present paper deals with vibration analysis, the ideas discussed pertain to buckling problems and static response problems as well. Other generalizations of the membrane results can be made by analogy with vibrations or buckling of polygonal plates. For example, it has been shown previously by Conway[10] that the vibrating membrane problem is directly analogous to the vibration of a polygonal plate with simply supported boundaries. And, it is also well known that the vibrating plate equation is analogous to the buckling of a plate under (isotropic) hydrostatic compression in the plane of the plate.[10] No discussion of multi-symmetric structural analysis would be complete without some mention of the cyclic symmetry capability in NASTRAN[13] or the 1956 attempt by Melvin and Edwards[14] to bring group theory to the attention of structural dynamicists.

By approaching the problem of symmetric structures using the results of symmetrical electrical components[15] the developers of NASTRAN have automated and generalized much of what has been discussed herein. The approach of Ref. 13 centers on the analysis of large, general symmetric structures.

Melvin and Edwards' paper[14] is written in terms most familiar to workers in physics and quantum mechanics. Although it is an interesting paper and discusses group theory applied to vibrations of membranes and plates, apparently the jargon in which it is written caused it to be neglected by most structural dynamicists. Perhaps the present paper will encourage dynamicsts to re-examine Melvin and Edwards' work and obtain a greater understanding of multi-symmetric structural analysis. Other related works which the interested reader may find useful are Refs. 16-21.

Conclusions

The primary conclusion of this report is that symmetry concepts can be used to reduce the size of the eigenvalue problem which must be modeled and solved using a discrete formulation. A second conclusion is that an understanding of group theory can be used to classify and order the vibration modes of multisymmetric structures. This classification has been demonstrated for two simple examples, namely a triangle and a hexagon.

The author cannot take claim for having discovered either of the preceding two conclusions, since it is clear that both of these items have been known to physicists ever since the early 1930's. It is equally apparent that these ideas have not been appreciated by workers in structural dynamics.

The primary new innovation of this report is that it is not necessary to forumlate a model for more than a small pie-shaped segment of the structure. This innovation required the development of appropriate boundary conditions on the edges of the pie-shaped segment, and these have been derived for a triangle and a hexagon. These idas (which have been demonstrated for flat equilateral polygonal membranes) can be generalized to "real-world" three-dimensional structures, as demonstrated in Ref. 13.

Finally, from a practical standpoint, these ideas of symmetry and their application can significantly reduce the dimension (i.e., size) of the eigenvalue problem which must be solved to compute the vibration modes of multi-symmetric structures.

Appendix A: Boundary Conditions: Equilateral Triangular Membrane

In the main text, it was noted that certain vibration modes of an equilateral triangular membrane are degenerate. In particular, the modes denoted by ϕ, $C_3\phi$, and $\bar{C}_3\phi$, shown in Fig. 8) are all termed "companion modes," since they have exactly the same eigenvalue.

The purpose of this section is to provide a derivation of the relationships

$$\phi = -(C_3\phi + \bar{C}_3\phi) \tag{A1}$$

$$\psi = 1/\sqrt{3}(\bar{C}_3\phi - C_3\phi) \tag{A2}$$

referred to previously, as well as the boundary conditions

$$\phi + \sqrt{3}\psi = 0 \tag{A3}$$

and

$$\nabla(\psi - \sqrt{3}\phi)\cdot n = 0 \tag{A4}$$

given previously.

To begin this development, refer to the mode labeled "ϕ" in Fig. 8. If we "rotate the mode" clockwise by $2\pi/3$ radians (120°) while keeping our (x,y) coordinate axes fixed, we obtain the mode labelled "$C_3\phi$" in Fig. 8b. Similarly, if we rotate the mode counter-clockwise by $2\pi/3$, we obtain the mode $\bar{C}_3\phi$ in Fig. 8c. If we wish, we can think of the symbols C_3 and $\bar{C}_3$ as "operators" which take the mode ϕ and transform it by a rotation.

Now it is clear that ϕ, $C_3\phi$, and $\bar{C}_3\phi$ all have the same eigenvalue and are "equally qualified candidates" to be chosen as a vibration mode. From previous work done in group theory (Ref. 7) we know that only *two* of these three companion modes are independent. Referring to the (vertical) y-axis in Fig. 8, we see that we can form an antisymmetric mode by taking the sum

$$\phi = -(\bar{C}_3\phi + C_3\phi) \tag{A5}$$

and a symmetric mode by taking the difference

$$\psi = k(\bar{C}_3\phi - C_3\phi) \tag{A6}$$

where k is a constant yet to be determined. Operating on both sides of Eq. (A5) with the "rotation operator" C_3, we have

$$C_3\phi = -[C_3\bar{C}_3\phi + C_3C_3\phi] \tag{A7}$$

but, from the "multiplication table" of group operations (Ref. 7) we have

$$C_3\bar{C}_3\phi \equiv \phi \tag{A8}$$

and

$$C_3C_3\phi = C_3^2\phi = \bar{C}_3\phi \tag{A9}$$

Substituting Eqs. (A8) and (A9) into Eq. (A7) gives

$$C_3\phi = -(\phi + \bar{C}_3\phi) \tag{A10a}$$

and by operating with $\bar{C}_3$ in a similar fashion on Eq. (A5) gives the analogous relation

$$\bar{C}_3\phi = -(C_3\phi + \phi) \tag{A10b}$$

Squaring both sides of Eq. (A10a) and integrating over the area of the triangle gives

$$\iint (C_3\phi)^2 dS = \iint \phi^2 dS + 2\iint \phi(\bar{C}_3\phi)dS + \iint (\bar{C}_3\phi)^2 dS \tag{A11}$$

which yield the results that

$$2\int \phi(\bar{C}_3\phi)dS = -1 \tag{A11}$$

where we have chosen to normalize the mode ϕ such that

$$\iint \phi^2 dS = 1 \tag{A12a}$$

where the integration is performed over the entire equilateral triangle. The modes $C_3\phi$ and $\bar{C}_3\phi$ are just "rotated versions" of ϕ, and they satisfy the same normalization:

$$\iint (C_3\phi)^2 dS = 1 \tag{A12b}$$

$$\iint (\bar{C}_3\phi)^2 dS = 1 \tag{A12c}$$

The normalization for ψ is chosen in the same fashion

$$\iint \psi^2 dS = 1 \tag{A13}$$

The constant k is determined by using Eqs. (A12) and (A13) as follows. Squaring both sides of Eq. (A6) gives

$$\psi^2 = k^2[(\bar{C}_3\phi)^2 - 2(C_3\phi)(\bar{C}_3\phi) + (C_3\phi)^2] \tag{A14a}$$

Integrating both sides of Eq. (A14a) over the area S of the equilateral triangle gives

$$\iint \psi^2 dS = 1 = k^2[1 - 2\iint (C_3\phi)(\bar{C}_3\phi)dS + 1] \tag{A14b}$$

where Eqs. (A12) and (A13) have been used.

One can show that

$$-2\iint (C_3\phi)(\bar{C}_3\phi)dS = 1$$

in analogy with Eq. (A11) given previously. Thus, Eq. (A14a) reduces to

$$3k^2 = 1$$

whence

$$k = 1/\sqrt{3}$$

is the normalizing factor for the ψ mode. Thus, with

$$k = 1/\sqrt{3}$$

Eqs. (A5) and (A6) give

$$\phi = -(\bar{C}_3\phi + C_3\phi) \tag{A15a}$$

and

$$\psi = 1/\sqrt{3}(\bar{C}_3\phi - C_3\phi) \tag{A15b}$$

which were the relations used previously in the main text. Now, using Eqs. (A15) and adding, we have

$$\phi + \sqrt{3}\psi = -2C_3\phi \tag{A16}$$

Referring to Fig. 8b, we see that the mode $C_3\phi$ is identically zero along the right-running diagonal. Thus, Eq. (A16) provides one of the desired boundary conditions on ϕ and ψ namely

$$\phi + \sqrt{3}\psi = 0$$

along the hypotenuse of the 30-60-90 triangle in Fig. 9.

Now, referring back to Fig. 8 where we have the mode

$$\psi = 1/\sqrt{3}(\bar{C}_3\phi - C_3\phi)$$

we note that ψ is symmetric, such that $\partial\psi/\partial n = 0$ along the (vertical) y-axis.

Just as the mode ϕ was "rotated" to give modes $C_3\phi$ and $\bar{C}_3\phi$, we can do the same operations on ψ to give

$$C_3\psi = 1/\sqrt{3}(C_3\bar{C}_3\phi - C_3^2\phi) \tag{A17a}$$

$$C_3\psi = 1/\sqrt{3}(\phi - \bar{C}_3\phi) \tag{A17b}$$

The modes ψ, $C_3\psi$, and $\bar{C}_3\psi$ are shown schematically in Fig. 16. Referring to Fig. 16, we see that the mode $C_3\psi$ is symmetric along the right-running diagonal (i.e., the hypotenuse of the 30-60-90 triangle in Fig. 9).

Thus, differentiating both sides of Eq. (A17b), we have

$$\frac{\partial}{\partial n}(C_3\psi) = 0 = \frac{1}{\sqrt{3}}\left[\frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}(\bar{C}_3\phi)\right] \tag{A18}$$

Hence, along this hypotenuse, we have

$$\frac{\partial\phi}{\partial n} = \frac{\partial}{\partial n}(\bar{C}_3\phi) \tag{A19}$$

But, we also have the general relation

$$\phi = -(C_3\phi + \bar{C}_3\phi)$$

and differentiating both sides, we have

$$\frac{\partial\phi}{\partial n} = -\frac{\partial}{\partial n}(C_3\phi) - \frac{\partial}{\partial n}(\bar{C}_3\phi) \tag{A20}$$

which is valid all over the equilateral triangle, and along the right-running diagonal in particular.

Combining Eqs. (A19) and (A20) gives

$$\frac{\partial\phi}{\partial n} = \frac{\partial}{\partial n}(\bar{C}_3\phi) = -\frac{\partial}{\partial n}(C_3\phi) - \frac{\partial}{\partial n}(C_3\phi) - \frac{\partial}{\partial n}(\bar{C}_3\phi)$$

which reduces to

$$\frac{\partial}{\partial n}(C_3\phi) = -2\frac{\partial}{\partial n}(\bar{C}_3\phi) \tag{A21}$$

Equation (A21) holds along the right-running diagonal.

Finally, differentiating Eq. (A2) for ψ gives

$$\frac{\partial\psi}{\partial n} = \frac{1}{\sqrt{3}}\left[\frac{\partial}{\partial n}(\bar{C}_3\phi) - \frac{\partial}{\partial n}(C_3\phi)\right] \tag{A22}$$

in general, and using Eq. (A21) in (A22) we have

$$\frac{\partial\psi}{\partial n} = \frac{1}{\sqrt{3}}\left[\frac{\partial}{\partial n}(\bar{C}_3\phi) + 2\frac{\partial}{\partial n}(\bar{C}_3\phi)\right] \tag{A23}$$

$$\frac{\partial\psi}{\partial n}\quad \frac{1}{\sqrt{3}}\,3\,\frac{\partial}{\partial n}(C_3\phi) \tag{A24}$$

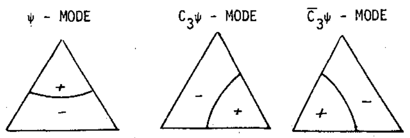

Fig. 16 Degenerate modes ψ, $C_3\psi$, and $\bar{C}_3\psi$.

but

$$\frac{\partial(C_3\phi)}{\partial n}=\frac{\partial\phi}{\partial n}$$

from Eq. (A19), so, we have, finally

$$\frac{\partial\psi}{\partial n}=\sqrt{3}\,\frac{\partial\phi}{\partial n} \tag{A25}$$

as the other boundary condition along the hypotenuse.

In another form, we can summarize our results as

$$\phi+\sqrt{3}\psi=0$$

and

$$\nabla(\psi-\sqrt{3}\phi)\cdot n=0$$

which are the required boundary conditions for ϕ and ψ along the right-running diagonal (hypotenuse) of Fig. 9.

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